**Semester Two Examination, 2019**

**Question/Answer booklet**

# MATHEMATICS SPECIALIST UNITS 3 AND 4

**Section Two: Calculator-assumed**

# SOLUTIONS

Student number: In figures

In words ______________________________

Your name ______________________________

## Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

## Materials required/recommended for this section

***To be provided by the supervisor***
This Question/Answer booklet
Formula sheet (retained from Section One)

***To be provided by the candidate***
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

## Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

## Structure of this paper

| Section | Number of questions available | Number of questions to be answered | Working time (minutes) | Marks available | Percentage of examination |
|---|---|---|---|---|---|
| Section One: Calculator-free | 8 | 8 | 50 | 52 | 35 |
| Section Two: Calculator-assumed | 13 | 13 | 100 | 98 | 65 |
| | | | | **Total** | 100 |

## Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

**Section Two: Calculator-assumed** **65% (98 Marks)**

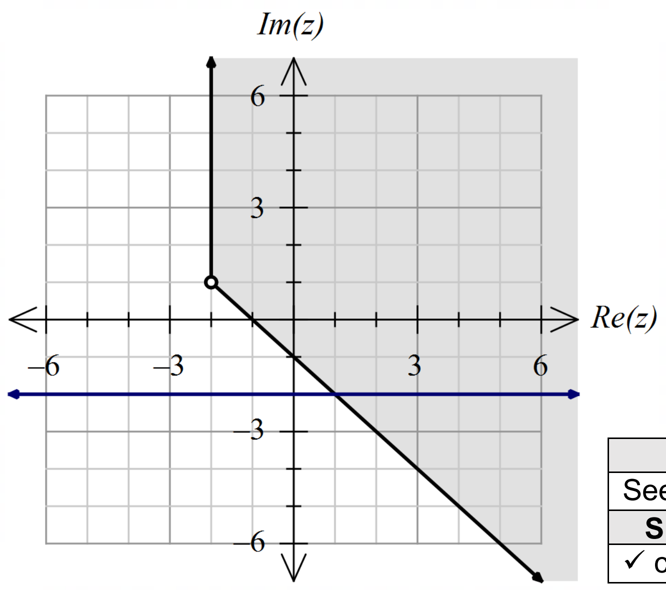
This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

---

**Question 9** **(4 marks)**

The locus of a complex number $z$ is shown below.




| Solution (b) |
|---|
| See diagram |
| **Specific behaviours** |
| ✓ correct line |

(a) Without using $\mathrm{Re}(z)$ or $\mathrm{Im}(z)$, write an inequality in terms of $z$ for the locus. (3 marks)

| Solution |
|---|
| $-\frac{\pi}{4} \leq \arg\big(z - (-2 + i)\big) \leq \frac{\pi}{2}$ |
| **Specific behaviours** |
| ✓ lower argument<br>✓ upper argument<br>✓ translation |

(b) Add the locus for $|z| = |z + 4i|$ to the diagram above. (1 mark)

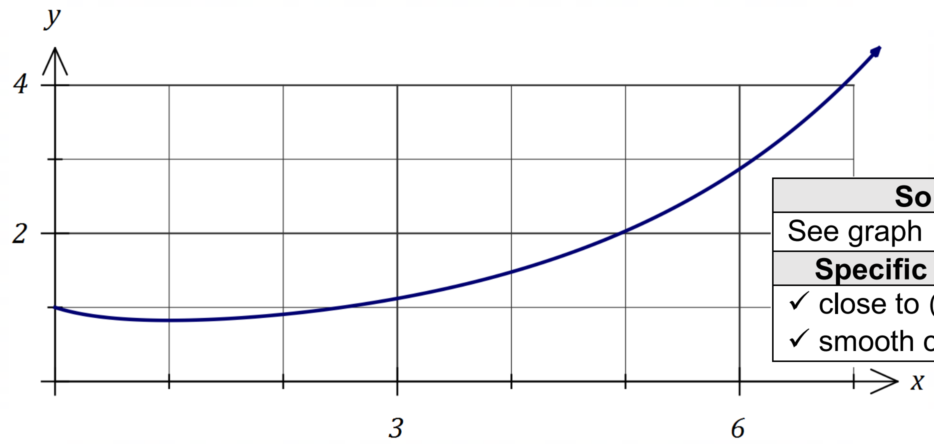
**Question 10** **(4 marks)**

Solve the equation $z^5 = -16 + 16\sqrt{3}i$, giving solutions in polar form $r\,\text{cis}\,\theta$ where $-\pi < \theta \leq \pi$ and $r > 0$.

| Solution |
|---|
| $z^5 = 32\,\text{cis}\left(\frac{2\pi}{3}\right)$ <br> $z_k = \sqrt[5]{32}\,\text{cis}\left(\frac{2\pi}{3} \times \frac{1}{5} + \frac{2n\pi}{5}\right)$ <br> $z_k = 2\,\text{cis}\left(\frac{2\pi}{15} + \frac{2n\pi}{5}\right), n = -2, -1, 0, 1, 2$ <br> $z_1 = 2\,\text{cis}\left(-\frac{2\pi}{3}\right)$ <br> $z_2 = 2\,\text{cis}\left(-\frac{4\pi}{15}\right)$ <br> $z_3 = 2\,\text{cis}\left(\frac{2\pi}{15}\right)$ <br> $z_4 = 2\,\text{cis}\left(\frac{8\pi}{15}\right)$ <br> $z_5 = 2\,\text{cis}\left(\frac{14\pi}{15}\right)$ |
| **Specific behaviours** |
| ✓ converts to polar form <br> ✓ applies de Moivre's <br> ✓ one correct root <br> ✓ all correct roots |

**Question 11** **(7 marks)**

(a) Sketch the graph of $y = \dfrac{e^{0.5x}}{1+x}$ on the axes below. (2 marks)




| Solution |
|---|
| See graph |
| **Specific behaviours** |
| ✓ close to $(0,1)$ & $(6,3)$ |
| ✓ smooth curve |

The Trapezoidal Rule can be used to determine the numerical approximation of a definite integral when an antiderivative cannot be found. When a continuous interval $[a_0,\ a_n]$ is divided into $n$ smaller intervals of equal width $w$, the bounds of these smaller intervals can be denoted by $a_0,\ a_1,\ a_2, \dots,\ a_{n-1},\ a_n$. The Trapezoidal Rule is then expressed as follows:

$$\int_{a_0}^{a_n} f(x)\,dx = \frac{w}{2}\left[f(a_0) + 2f(a_1) + 2f(a_2) + \cdots + 2f(a_{n-1}) + f(a_n)\right]$$

(b) Use the above rule to determine an estimate to 4 decimal places for $\displaystyle\int_0^6 \frac{e^{0.5x}}{1+x}\,dx$ using

(i) 2 intervals. (2 marks)

| Solution |
|---|
| $\frac{3}{2}\big(f(0) + 2f(3) + f(6)\big) = 9.1653$ |
| **Specific behaviours** |
| ✓ indicates use of rule with correct intervals |
| ✓ correct estimate, to 4 dp |

(ii) 24 intervals. (3 marks)

| Solution |
|---|
| $E = \frac{0.25}{2}\big(f(0) + 2f(0.25) + \cdots + 2f(5.75) + f(6)\big)$ <br> $\approx 8.1786$ |
| **Specific behaviours** |
| ✓ indicates use of correct intervals |
| ✓ indicates use of rule |
| ✓ correct estimate, to 4 dp |

| CAS |
|---|
| Define $f(x)=\frac{e^{0.5x}}{1+x}$ |
| done |
| $\frac{3}{2}(f(0)+2f(3)+f(6))$ |
| 9.165310429 |
| $\frac{0.25}{2}\left(\sum_{n=1}^{23}(2f(0.25n))+f(0)+f(6)\right)$ |
| 8.178577297 |

**Question 12** **(8 marks)**

The diameter of copper wire produced by a machine is normally distributed with a mean of $485\ \mu\text{m}$ and a variance of $396\ \mu\text{m}^2$.

A production supervisor routinely takes a random sample of $45$ diameters and calculates their mean, $\bar{X}$.

(a) Describe the distribution of $\bar{X}$. (3 marks)

| Solution |
|---|
| $\bar{X}$ is normally distributed<br>$396 \div 45 = 8.8$<br>$\bar{X} \sim N(485, 8.8) \sim N(485, 2.966^2)$ |
| **Specific behaviours** |
| ✓ states normally distributed<br>✓ states mean<br>✓ states variance (or sd) |

(b) Determine the probability that the mean of a random sample of $45$ diameters is greater than $490\ \mu\text{m}$. (1 mark)

| Solution |
|---|
| $P(\bar{X} > 490) = 0.0459$ |
| **Specific behaviours** |
| ✓ correct probability |

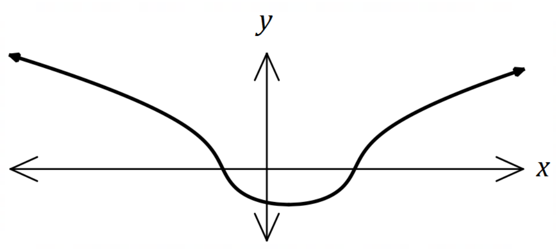
(c) Repeated random sampling of $n$ diameters from the machine shows that there is a $17\%$ chance that the sample mean is less than $482\ \mu\text{m}$. Determine $n$. (4 marks)

| Solution 1 |
|---|
| $\bar{Y} \sim N\left(485, \frac{396}{n}\right)$<br>$P(\bar{Y} < 482) = 0.17$<br>$\sqrt{\frac{396}{n}} = 3.144$<br>$n = 40$ |
| **Specific behaviours** |
| ✓ writes distribution<br>✓ writes probability statement<br>✓ writes equation for $n$<br>✓ correct value of $n$ |

| Solution 2 |
|---|
| $\bar{Y} \sim N\left(485, \frac{396}{n}\right)$<br>$P(Z < k) = 0.17 \Rightarrow k = -0.9542$<br>$\frac{482 - 485}{\sqrt{396 \div n}} = -0.9542$<br>$n = 40$ |
| **Specific behaviours** |
| ✓ writes distribution<br>✓ indicates use of $z$-score, $k$<br>✓ writes equation for $n$<br>✓ correct value of $n$ |

**Question 13** **(6 marks)**

A particle is moving along the curve shown below with equation $x^2 - 2x = y^3 + 3y + 8$.




The $x$-coordinate of the particle is changing at a constant rate given by $\frac{dx}{dt} = -4$.

Determine the rates at which the $y$-coordinate of the particle is changing when $y = 0$.

**Solution**

$$2x - 2 = 3y^2\frac{dy}{dx} + 3\frac{dy}{dx}$$

$$y = 0 \Rightarrow \frac{dy}{dx} = \frac{2x-2}{3}$$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$
$$= \frac{-4(2x-2)}{3}$$

$$y = 0 \Rightarrow x^2 - 2x - 8 = 0$$
$$(x+2)(x-4) = 0$$

$$\text{At } (-2, 0), \frac{dy}{dt} = \frac{-4(-6)}{3} = 8$$

$$\text{At } (4, 0), \frac{dy}{dt} = \frac{-4(6)}{3} = -8$$

**Specific behaviours**

- ✓ implicitly differentiates wrt $x$
- ✓ expression for $dy/dx$
- ✓ expression for $dy/dt$
- ✓ determines $x$ when $y = 0$
- ✓ one correct rate
- ✓ both correct rates

**Alternative part solution**

$$2x\frac{dx}{dt} - 2\frac{dx}{dt} = 3y^2\frac{dy}{dt} + 3\frac{dy}{dt}$$

$$y = 0, \quad \frac{dx}{dt} = -4$$

$$\frac{dy}{dt} = \frac{-4(2x-2)}{3}$$

**Specific behaviours**

- ✓ implicitly differentiates LHS wrt $t$
- ✓ implicitly differentiates RHS wrt $t$
- ✓ expression for $dy/dt$

**Question 14** **(7 marks)**

Water, containing $8$ grams of dissolved sugar per litre, flows into a tank at a constant rate of $15$ litres per hour.

Water is drawn from the tank, initially containing $300$ litres of water with no dissolved sugar, at the same constant rate of $15$ litres per hour.

Let the weight of sugar in the tank after $t$ hours be $w$ grams and assume that the sugar is always evenly dissolved throughout the water in the tank.

(a) By considering the rate at which dissolved sugar flows in and out of the tank, show that (2 marks)

$$\frac{dw}{dt}=\frac{2400-w}{20}.$$

| Solution |
| --- |
| Rate out: $15\times\frac{w}{300}=\frac{w}{20}$<br><br>Rate in: $15\times 8=120$<br><br>$\frac{dw}{dt}=120-\frac{w}{20}=\frac{2400-w}{20}$ |
| **Specific behaviours** |
| ✓ derives rate out<br>✓ derives rate in and uses difference |

The water drawn from the tank can be used in a manufacturing process once the level of dissolved sugar exceeds $1.5$ grams per litre.

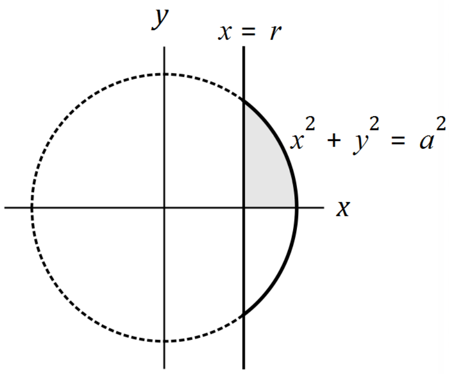
(b) Derive an equation for $w$ in terms of $t$ and hence determine how long this will take. (5 marks)

| Solution |
| --- |
| $\int\frac{1}{2400-w}dw=\int\frac{1}{20}dt$<br><br>$-\ln(2400-w)=\frac{t}{20}+c$<br><br>$2400-w=ke^{-0.05t}$<br><br>$t=0, w=0\Rightarrow k=2400$<br>$w=2400(1-e^{-0.05t})$<br><br>$1.5\times 300=2400(1-e^{-0.05t})$<br>$t=4.153\text{ h}$<br>Will take $4$ hours $9$ minutes. |
| **Specific behaviours** |
| ✓ separates variables<br>✓ integrates both sides, shows constant<br>✓ eliminates logs<br>✓ correct equation<br>✓ solves for $t$ |

**Question 15** **(8 marks)**

A manufacturer drills a hole of radius $r$ through the centre of a solid sphere of radius $a$.

To determine the volume $V$ of the remaining solid, consider the circle with equation $x^2 + y^2 = a^2$ and the line with equation $x = r$, as shown in the diagram below.




(a) Determine the coordinates of the point of intersection of the line and the circle in the first quadrant in terms of $r$ and $a$. (2 marks)

| Solution |
|---|
| $x = r \Rightarrow y = \sqrt{a^2 - r^2}$<br><br>Coordinates: $\left(r, \sqrt{a^2 - r^2}\right)$. |
| **Specific behaviours** |
| ✓ expresses $y$ in terms of $a$ and $r$<br>✓ coordinates of point |

(b) Construct a single integral to determine the volume of revolution obtained when the shaded region in the diagram above is rotated about the $y$-axis. (2 marks)

| Solution |
|---|
| $I = \pi \int_0^{\sqrt{a^2-r^2}} (a^2 - y^2) - (r^2)\, dy$<br>$= \pi \int_0^{\sqrt{a^2-r^2}} (a^2 - r^2) - y^2\, dy$ |
| **Specific behaviours** |
| ✓ difference of squares of relations<br>✓ correct integral, with limits |

(c) Evaluate your integral from (b) and hence determine a formula for $V$, the volume of the remaining solid. (3 marks)

| Solution |
|---|
| $$V_H = \pi\left[(a^2-r^2)y - \frac{y^3}{3}\right]_0^{\sqrt{a^2-r^2}}$$ $$= \pi\left[(a-r^2)\sqrt{a^2-r^2} - \frac{\sqrt{a^2-r^2}^3}{3}\right] - [(a^2-r^2)(0)-0]$$ $$= \pi\left[(a^2-r^2)^{\frac{3}{2}} - \frac{(a^2-r^2)^{\frac{3}{2}}}{3}\right]$$ $$= \frac{2\pi\sqrt{(a^2-r^2)^3}}{3}$$ $$V = \frac{4\pi\sqrt{(a^2-r^2)^3}}{3}$$ |
| **Specific behaviours** |
| ✓ shows correct antiderivative<br>✓ substitutes and simplifies $V_H$ (hemisphere volume)<br>✓ doubles to obtain correct formula |

(d) Hence, or otherwise, determine the exact volume of the remaining solid when a hole of radius 8 cm is drilled through the centre of a solid sphere of radius 17 cm. (1 mark)

| Solution |
|---|
| $$V = \frac{4\sqrt{(17^2-8^2)^3}}{3}$$ $$= 4500\pi$$ |
| **Specific behaviours** |
| ✓ correct volume (allow ft from (c)) |

**Question 16** **(8 marks)**

A researcher used data from a sample of $162$ newborn babies in order to estimate the mean weight and length of newborns in a large city.

(a) The weights of the babies in the sample had a mean of $3.35$ kg and a standard deviation of $0.58$ kg.

(i) Use this data to obtain a $90\%$ confidence interval for the mean weight of a newborn baby in the city. (2 marks)

| Solution |
|---|
| $(3.275, 3.425)$ |
| **Specific behaviours** |
| ✓ lower bound<br>✓ upper bound |

(ii) State two assumptions made when constructing your confidence interval. (2 marks)

| Solution |
|---|
| Sample was obtained randomly<br>Sample values are independent of each other<br>Sample means are normally distributed |
| **Specific behaviours** |
| ✓ first assumption<br>✓ second assumption |

(b) The $95\%$ confidence interval for the mean length $L$ cm of newborn babies derived from the sample was $(50.82, 51.58)$. Determine the sample mean and standard deviation used to construct this interval. (4 marks)

| Solution |
|---|
| $E = \dfrac{51.58 - 50.82}{2} = 0.38$<br><br>$\bar{x} = 50.82 + 0.38 = 51.2 \text{ cm}$<br><br>$1.96 \times \dfrac{s}{\sqrt{162}} = 0.38$<br><br>$s = 2.468 \text{ cm}$ |
| **Specific behaviours** |
| ✓ margin of error<br>✓ mean<br>✓ equation for sd<br>✓ standard deviation |

**Question 17** **(6 marks)**

(a) State the equations of all asymptotes of the graph of $y = \dfrac{8x^2 - 75}{x^2 - 25}$. (2 marks)

| Solution |
|---|
| $y = 8$<br><br>$x = 5, \qquad x = -5$ |
| **Specific behaviours** |
| ✓ horizontal asymptote<br>✓ both vertical asymptotes |

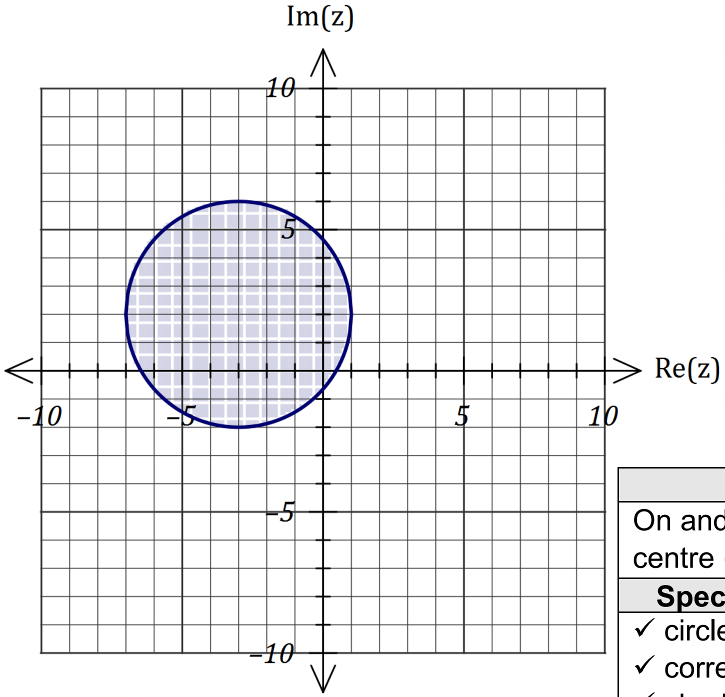
(b) Let $f(x) = \dfrac{ax^2 + bx + c}{x + d}$.

The graph of $y = f(x)$ has no roots, a $y$-intercept of $4$ and two asymptotes (with equations $x = -2$ and $y = 3x - 1$). Determine the value of each of the constants $a, b, c$ and $d$. (4 marks)

| Solution |
|---|
| $y = 3x - 1 + \dfrac{k}{x+2}$<br><br>$x = 0, y = 4 \Rightarrow 4 = -1 + \dfrac{k}{2} \Rightarrow k = 10$<br><br>$y = 3x - 1 + \dfrac{10}{x+2} = \dfrac{(3x-1)(x+2) + 10}{x+2} = \dfrac{3x^2 + 5x + 8}{x+2}$<br><br>$a = 3, \qquad b = 5, \qquad c = 8, \qquad d = 2$ |
| **Specific behaviours** |
| ✓ uses asymptotes to form equation<br>✓ uses intercept to determine $k$<br>✓ combines into single fraction<br>✓ correctly states all four values<br>*(Max ✓✓ if $b$ incorrect)* |

**Question 18** **(8 marks)**

(a) Indicate the subset of points in the complex plane that satisfy $|z + 3 - 2i| \leq 4$ on the axes below. (3 marks)




| Solution |
|---|
| On and inside circle with centre $(-3, 2)$ and $r = 4$. |
| **Specific behaviours** |
| ✓ circle, correct centre<br>✓ correct radius<br>✓ shades region |

(b) Given that $|z + 3 - 2i| \leq 4$, determine

(i) the maximum value of $\text{Im}(z)$. (1 mark)

| Solution |
|---|
| Maximum is $6$. |
| **Specific behaviours** |
| ✓ correct value |

(ii) the minimum value of $|z - 2|$. (2 marks)

| Solution |
|---|
| Require minimum distance from $(2, 0)$ to point on circle - will lie on ray from centre.<br>$\sqrt{5^2 + 2^2} - 4 = \sqrt{29} - 4 \approx 1.39$ |
| **Specific behaviours** |
| ✓ indicates correct method<br>✓ correct value |

(iii) the minimum value of $\text{Re}(iz)$. (2 marks)

| Solution |
|---|
| $\text{Re}(iz) = -\text{Im}(z)$<br>Hence minimum value is $-6$. |
| **Specific behaviours** |
| ✓ indicates correct method<br>✓ correct value |

**Question 19** **(12 marks)**

Points $A$, $B$ and $C$ lie in plane $\Pi$ and have position vectors $\begin{pmatrix}-2\\0\\9\end{pmatrix}$, $\begin{pmatrix}3\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\2\\4\end{pmatrix}$ respectively.

Point $B$ also lies on the sphere $S$ that has centre $A$.

(a) Determine the vector equation of $S$. (3 marks)

| Solution |
|---|
| $\overrightarrow{AB} = \begin{pmatrix}3\\-1\\1\end{pmatrix} - \begin{pmatrix}-2\\0\\9\end{pmatrix} = \begin{pmatrix}5\\-1\\-8\end{pmatrix}$ <br> $\lvert\overrightarrow{AB}\rvert = 3\sqrt{10}$ <br> $\left\lvert \mathbf{r} - \begin{pmatrix}-2\\0\\9\end{pmatrix}\right\rvert = 3\sqrt{10}$ |
| **Specific behaviours** |
| ✓ vector $\overrightarrow{AB}$ <br> ✓ calculates radius <br> ✓ states vector equation |

(b) Determine the Cartesian equation for plane $\Pi$. (4 marks)

| Solution |
|---|
| $\overrightarrow{AC} = \begin{pmatrix}2\\2\\4\end{pmatrix} - \begin{pmatrix}-2\\0\\9\end{pmatrix} = \begin{pmatrix}4\\2\\-5\end{pmatrix}$ <br> $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}21\\-7\\14\end{pmatrix}$ <br> $\mathbf{n} = \begin{pmatrix}3\\-1\\2\end{pmatrix}$ <br> $\overrightarrow{OA} \cdot \mathbf{n} = \begin{pmatrix}-2\\0\\9\end{pmatrix} \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix} = 12$ <br> $3x - y + 2z = 12$ |
| **Specific behaviours** |
| ✓ vector $\overrightarrow{AC}$ or $\overrightarrow{BC}$, etc <br> ✓ uses cross product to obtain normal <br> ✓ uses dot product to obtain constant <br> ✓ states Cartesian form |

A point and its reflection in a plane are equidistant from the plane and lie on a line that is perpendicular to the plane.

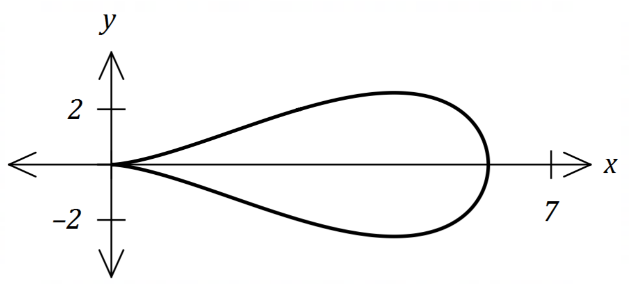
Point $P$ has position vector $\begin{pmatrix} 9 \\ -9 \\ 16 \end{pmatrix}$.

(c) Determine the position vector of $P'$, the reflection of $P$ in plane $\Pi$. (5 marks)

| Solution |
|---|
| Line through $P$ perpendicular to plane: $$\mathbf{r} = \begin{pmatrix} 9 \\ -9 \\ 16 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$ Line intersects plane at $Q$, when: $$\left[\begin{pmatrix} 9 \\ -9 \\ 16 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}\right] \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 12$$ $$3(9 + 3\lambda) - (-9 - \lambda) + 2(16 + 2\lambda) = 12$$ $$\lambda = -4$$ Hence $\overrightarrow{OQ} = \mathbf{r}(-4)$ and so $\overrightarrow{OP'} = \mathbf{r}(-8)$: $$\overrightarrow{OP'} = \begin{pmatrix} 9 \\ -9 \\ 16 \end{pmatrix} - 8 \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -15 \\ -1 \\ 0 \end{pmatrix}$$ |
| **Specific behaviours** |
| ✓ writes equation of line through $P$<br>✓ substitutes line into plane<br>✓ solves for $\lambda$<br>✓ indicates method to determine $P'$<br>✓ correct position vector for $P'$ |

**Question 20** **(11 marks)**

The path of a particle is shown in the diagram below.




The position vector of the particle after $t$ seconds is given by $\mathbf{r}(t) = \begin{pmatrix} 3 - 3\sin t \\ 2\cos t - \sin 2t \end{pmatrix}$ centimetres, for $t \geq 0$.

(a) Determine the initial position of the particle. (1 mark)

| Solution |
|---|
| $\mathbf{r}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ cm |
| **Specific behaviours** |
| ✓ correct position |

(b) Determine the acceleration vector of the particle at the instant it first reaches the origin. (3 marks)

| Solution |
|---|
| $x = 0 \Rightarrow 3 - 3\sin t = 0 \Rightarrow t = \dfrac{\pi}{2}$<br><br>$\mathbf{v}(t) = \begin{pmatrix} -3\cos t \\ -2\sin t - 2\cos 2t \end{pmatrix}$<br><br>$\mathbf{a}(t) = \begin{pmatrix} 3\sin t \\ -2\cos t + 4\sin 2t \end{pmatrix}$<br><br>$\mathbf{a}\left(\dfrac{\pi}{2}\right) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ cm s$^{-2}$ |
| **Specific behaviours** |
| ✓ indicates time<br>✓ differentiates twice<br>✓ substitutes and simplifies |

(c) Determine the distance travelled by the particle from the time it leaves its initial position until the time it first reaches the origin. (3 marks)

| Solution |
|---|
| $$\lvert\mathbf{v}(t)\rvert = \sqrt{(-3\cos t)^2 + (-2\sin t - 2\cos 2t)^2}$$ $$s = \int_0^{\frac{\pi}{2}} \lvert\mathbf{v}(t)\rvert\, dt$$ $$s \approx 3.629 \text{ cm}$$ |
| **Specific behaviours** |
| ✓ expression for magnitude of velocity<br>✓ forms correct integral<br>✓ correct distance |

(d) The Cartesian equation of the path of the particle is $ay^2 + bx^3 + 4x^4 = 0$. Determine the value of each of the constants $a$ and $b$. (4 marks)

| Solution |
|---|
| $$y = 2\cos t - 2\cos t \sin t$$ $$= 2\cos t\,(1 - \sin t)$$ $$x = 3 - 3\sin t \Rightarrow \frac{x}{3} = 1 - \sin t$$ $$y = 2\cos t\left(\frac{x}{3}\right)$$ $$3y = 2x\cos t$$ $$9y^2 = 4x^2\cos^2 t$$ $$= 4x^2(1 - \sin^2 t)$$ $$\sin t = \frac{x}{3} - 1 \Rightarrow \sin^2 t = \left(\frac{x}{3} - 1\right)^2 = \frac{x^2}{9} - \frac{2x}{3} + 1$$ $$9y^2 = 4x^2\left(1 - \frac{x^2}{9} + \frac{2x}{3} - 1\right)$$ $$9y^2 = \frac{8x^3}{3} - \frac{4x^4}{9}$$ $$81y^2 = 24x^3 - 4x^4 \Rightarrow a = 81, b = -24$$ |
| **Specific behaviours** |
| ✓ obtains relation between $x$, $y$ and $\cos t$<br>✓ obtains relation for $\sin^2 t$ in terms of $x$<br>✓ eliminates all trig terms<br>✓ correct values for constants |

**Question 21** **(9 marks)**

Researchers used a simulation to model the population of foxes $F(t)$ and the population of rabbits $R(t)$ on an island. The rates of change of each population after $t$ years are given by

$$\frac{dF}{dt} = 0.32R \text{ and } \frac{dR}{dt} = -0.08F.$$

(a) Briefly explain how the rabbit population is changing. (1 mark)

| Solution |
|---|
| Rabbit population is decreasing at a rate proportional to the population of foxes. |
| **Specific behaviours** |
| ✓ states decreasing |

(b) Show that $\frac{d^2R}{dt^2} = -0.0256R$. (2 marks)

| Solution |
|---|
| $\frac{d}{dt}\left(\frac{dR}{dt}\right) = \frac{d}{dt}(-0.08F)$ <br> $\frac{d^2R}{dt^2} = -0.08\frac{dF}{dt}$ <br> $= -0.08(0.32R)$ <br> $= -0.0256R$ |
| **Specific behaviours** |
| ✓ differentiates both sides <br> ✓ substitutes and simplifies |

The equation in part (b) suggests that a model of the form $R(t) = k\cos(at + b)$ would be appropriate, where $a, b$ and $k$ are positive constants.

(c) Explain this choice of model. (1 mark)

| Solution |
|---|
| The equation in (b) is of the form <br> $F'' = -a^2F$ <br> and represents simple harmonic motion. |
| **Specific behaviours** |
| ✓ indicates SHM |

The research model used $b = 0.27$ and the initial size of the rabbit population was $800$.

(d) Determine the value of $a$ and the value of $k$. (2 marks)

| Solution |
|---|
| $a^2 = 0.0256 \Rightarrow a = 0.16$<br><br>$800 = k\cos(0.27) \Rightarrow k = 830$ |
| **Specific behaviours** |
| ✓ value of $a$<br>✓ value of $k$ |

(e) Determine an equation for $F(t)$ in terms of $t$ and hence calculate the number of years it takes for the initial fox population to double in size. (3 marks)

| Solution |
|---|
| $\frac{dR}{dt} = -0.08F \Rightarrow F = -12.5\frac{dR}{dt}$<br><br>$R(t) = 830\cos(0.16t + 0.27)$<br>$\frac{dR}{dt} = -132.8\sin(0.16t + 0.27)$<br><br>$F(t) = (-12.5)(-132.8)\sin(0.16t + 0.27)$<br>$F(t) = 1660\sin(0.16t + 0.27)$<br><br>$F(0) = 443$<br>$F(t) = 2 \times 443 \Rightarrow t = 1.83$ years |
| **Specific behaviours** |
| ✓ obtains $R'(t)$<br>✓ obtains $F(t)$<br>✓ correct time to double |



**End of questions**

Supplementary page

Question number: _________

Supplementary page

Question number: _________

Supplementary page

Question number: _________

Supplementary page

Question number: _________